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Behavior of General One-Dimensional Diffusion Processes

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We develop simple rigorous techniques to estimate the behavior of general one-dimensional diffusion processes. Any one-dimensional diffusion process (with drift) can be mapped onto a symmetric diffusion through an explicit change of variable. For such processes we can estimate explicitly the diffusion exponent, the recurrence properties, and the large fluctuations. In a second part, we apply these results to different models (including the Sinaï random walk: diffusion in a random drift) and we show how the main features of the diffusion can be readily handled.

KEY WORDS: Diffusion processes; random walks; random walks in random environment.

1. INTRODUCTION

In this paper we consider general stationary one-dimensional diffusion processes on R defined by the stochastic equation

$$dX = \sigma(X) \cdot dB + \mu(X)\, dt \tag{1}$$

where B is the Brownian motion and σ is a space-dependent diffusion coefficient and μ is the drift. If we denote $P(0, x, t)$ the transition probability from 0 to x within time t, the stochastic equation (1) leads to the differential equation

$$1/2\, \partial^2/\partial x^2\, \sigma^2 P - \partial/\partial x\, \mu P = \partial P/\partial t \tag{2}$$

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Remark. For physicists, Eq. (2) is the Itô version of (1), while the Stratonovitch one would read

$$1/2\,\partial/\partial x\,\sigma\,\partial/\partial x\,\sigma P - \partial/\partial x\,\mu P = \partial P/\partial t \tag{3}$$

The choice of a version depends on the underlying physical situation. For instance, if $\mu = 0$, the Itô version in surely relevant when physically $\mathsf{E}(X(t))$ is constant.

We will provide some simple rigorous results in order to obtain the physical behavior of (2) depending on the shape of σ and μ. This behavior can usually be characterized by:

1. The diffusion exponent d_w. This is the exponent such that $|x_t|^{d_w}$ goes like t.

2. The asymptotic behavior (for large t) of $P(0, 0, t)$: the probability (with the above notations) to be at the origin at time t.

3. The asymptotic behavior (for large t) of $R(a, 0, t)$: the probability to be for the first time at 0 at time t starting from a [in the discrete case, one can define $R(0, 0, t)$, which is the distribution of recurrence times].

4. The large fluctuations as given by the tail of $P(0, x, t)$ for large x.

It is well known[1] that any one-dimensional diffusion process (even including a deterministic drift) can be mapped onto an Itô diffusion through a simple change of variable (see Section 3.3 for an example). Thus, in Section 2 rigorous bounds on these quantities will be established for Eq. (2) with $\mu = 0$. Our approach in this part is similar to that of ref. 2. In Section 3 we will show with more heuristic arguments how these bounds provide easily the behavior of the diffusion. We will study several examples.

The case where σ is a random ergodic variable (and $\mu = 0$) has already been studied in ref. 3 for arbitrary dimension: it is proved that $X(t)/\sqrt{t}$ converges in law to a Gaussian variable. In ref. 4 the same result is obtained for a one-dimensional discrete equivalent of (1). In either one-dimensional case one gets the exact value of the Gaussian variable which provides an effective diffusion coefficient σ_{eff}: $\sigma_{\text{eff}}^{-2} = \mathsf{E}(\sigma^{-2})$ (for higher dimension, the *Stratonovitch* case is difficult and has already been considered in the random case in ref. 3). We are interested in more general σ and μ. This concerns as well the deterministic cases where σ is inhomogeneous (for instance, σ behaves like a power at infinity) as random cases [$\mathsf{E}(\sigma^{-2})$ can be infinite] where we can study typical samples of σ. The first case occurs when one considers chaotic dynamical systems as diffusion processes with a speed going to 0 near the stability regions. In the random cases, contrary to previous works,[3-5] we do not attempt to get the convergence in law to some scaled process, but, as we can study typical samples, we can derive the behavior of some rare events (like the renewal events). As a last example, we consider random walks in random environments[5–7] (i.e., diffusion with random drifts).

The discrete equivalents of (2) and (3), are respectively, for $\mu = 0$,

$$dP_i/dt = -\lambda_i P_i + 1/2\lambda_{i+1} P_{i+1} + 1/2\lambda_{i-1} P_{i-1} \tag{2bis}$$

$$dP_i/dt = -1/2(\lambda_i + \lambda_{i+1}) P_i + 1/2\lambda_{i+1} P_{i+1} + 1/2\lambda_i P_{i-1} \tag{3bis}$$

where λ_i are the local transition rates. The discrete models are not studied there, but similar techniques could be used. In particular, the propositions of Section 2 can be readily translated.

2. GENERAL RESULTS

In this part, we assume that the diffusion process has been mapped onto an Itô diffusion ($\mu = 0$) through an appropriate change of variable. We are interested in diffusion without boundary points, so we suppose our processes are defined all over R. This requires that σ^{-2} is locally integrable. Furthermore, (1) makes sense only if σ is Lipshitz, but (2) always defines a diffusion. Thus, in the following we will consider general diffusion processes: symmetric Feller processes with absolutely continuous speed measure.[1] In fact, one can extend readily our results in the case where the diffusion has finite endpoints provided that these endpoints are open [b is open if $\int^b |b-x|\, \sigma^{-2}(x)\, dx = \infty$]. This case can occur when the Itô process comes from a process with drift defined all over R: the change of variable can map R onto a finite interval.

2.1. Behavior at Infinity

Let us first recall some features related to the behavior of σ at infinity. If σ^{-2} is integrable, then there exists a finite invariant measure $\sigma^{-2}\, dx$. Furthermore, in the Itô case the diffusion needs always an infinite time to reach $+\infty$ or $-\infty$. An odd behavior occurs when $|x|\, \sigma^{-2}$ is integrable near ∞: a particle comes from ∞ within a finite time. The infinite boundary point is then called *entrance*. Notice that in the Stratonovitch case the situation is quite different, since there never exists an invariant measure except if a particle can reach the infinity within a finite time (σ^{-1} integrable) in which case one has to specify suitable boundary conditions at infinity (this kind of diffusion is in fact easily mapped onto the Brownian motion on a finite interval).

2.2. Diffusion Velocity

The next physical property is the mean diffusion velocity, which is often related to a diffusion exponent d_w. Let us define

$$M(x) = 2\int_0^x dy \int_0^y dz\, \sigma^{-2}(z) \tag{4}$$

One easily checks that $M(0)=0$; $M(x)$ increases as $|x|$ increases and is unbounded even if there exists an invariant measure. Furthermore $M(X_t)-t$ is a martingale in the following sense[1]: if t^* is any Markovian time such that $\mathsf{E}(t^*)$ is finite and $M(X_t)$ is bounded for $t<t^*$, then

$$\mathsf{E}(M(X_{t^*})) = \mathsf{E}(t^*) \tag{5}$$

since $M(X)$ satisfies $(1/2\sigma^2\, \partial^2/\partial x + \partial/\partial t)[M(x)-t]=0$.

Thus, d_w may be fit using (5) with $t^* = \{$first time such that $M(X_t)=m\}$. For instance, if σ is such that $M(X)$ behaves at ∞ as $|X|^a$, then $d_w = a$ in the sense that the mean time to reach X behaves like $|X|^a$. On the other hand, we would like to know the value of $\mathsf{E}(|X_t|)$ and we could expect that it behaves like t^{1/d_w}: this is implicit in the definition of d_w in the physical literature. This is not generally true, even if σ is a "good" function of x (behaves like a power, for instance). In Section 2.5 we estimate $\mathsf{E}(|X_t|)$; however, at this step one can state the following result:

Proposition 1. If infinity is not an entrance boundary ($|x|\sigma^{-2}$ is not integrable at ∞), then the function M as defined in (4) satisfies

$$\mathsf{E}(M(X_t)) = t$$

Proof. Let us define a family of stopping times t_N:

$$t_N = \mathrm{Inf}(t, \text{ first time } \tau \text{ such that } |X_t| = N\}$$

and let c_N be the characteristic function of the event $\{t_N = t\}$. For the sake of simplicity, let us assume that σ (and consequently M) is symmetric. Then, for all N, (5) holds with $t^* = t_N$. Thus, we have

$$\mathsf{E}(c_N M(X_{t_N})) + \mathsf{E}(1-c_N)\, M(N) = \mathsf{E}(t_N) < t \tag{6}$$

As N goes to ∞, $\mathsf{E}(t_N)$ goes to t and the first term in (6) goes to $\mathsf{E}(M(X_t))$ by the monotone convergence theorem. Thus, (6) provides Proposition 1 if $\mathsf{E}((1-c_N)\, M(N))$ goes to 0. Let

$$M_2(x) = 2\int_0^x dy \int_0^y M(z)\, \sigma^{-2}(z)\, dz$$

Then $M_2(X_t) - 2\int_0^t M(X_s)\,ds$ is a martingale in the same sense that $M(X_t) - t$; in particular,

$$\mathsf{E}(c_N M_2(X_{t_N})) + \mathsf{E}(1-c_N)\, M_2(N) = 2\mathsf{E}\left(\int_0^{t_N} M(X_s)\,ds\right) < t^2 \tag{7}$$

since $\mathsf{E}(M(X_t)) < t$. Thus, $\mathsf{E}(1-c_N) < t^2/M_2(N)$ and if $K(N) = M(N)/M_2(N)$ goes to zero as N goes to ∞, then we have

$$\mathsf{E}(M(X_t)) = t \tag{8}$$

which fits the more usual definition of d_w. Let us now discuss the above condition and prove

$$K(N) \text{ goes to zero as } N \text{ goes to } \infty \Leftrightarrow \int_0^\infty \sigma^{-2}|x|\,dx = \infty \tag{9}$$

First, if there exists an invariant measure (i.e., $\int_0^\infty \sigma^{-2}\,dx < \infty$), then the proof is straightforward: for large x, $M(x) \approx C|x|$ and $M_2(x)$ behaves like $C'|x|$ if and only if $\int_0^\infty \sigma^{-2}|x|\,dx$ is finite, else $M_2(x) > C'x$ for any C'. Now, if $\int_0^\infty \sigma^{-2}(x)\,dx = \infty$, then the proof relies on the following lemma:

Lemma. Let f, g be positive increasing functions such that $f, g \to \infty$ as $x \to \infty$ and $f'/g' \to 0$ as $x \to \infty$, then f/g goes to zero as $x \to \infty$.

This lemma is obvious. We apply this lemma for $f = \int_0^x \sigma^{-2}\,dx$ and $g = \int_0^x \sigma^{-2} M(x)\,dx$; thus f, g satisfy the hypotheses, since $f'/g' = 1/M(x) \to 0$ as $x \to \infty$ and consequently $f/g \to 0$ as $x \to \infty$. Then, we can apply the lemma to $\int_0^x f$ and $\int_0^x g$ to get that $M(x)/M_2(x) \to 0$.

Now, if σ is not symmetric, the proof has to be readily modified: one has to consider separately the two limits at $+\infty$ or $-\infty$, and provided that K goes to 0 on both sides, (8) holds. This ends the proof of Proposition 1.

Remark. If K does not go to zero at ∞, then $M(X) \approx |X|$ and $M(X)$ is integrable with respect to the invariant measure. It is not surprising that (8) is false in this case and one expects that $\mathsf{E}(M(X_t))$ remains bounded.

2.3. Bounds on Laplace Transforms

This part mainly relies on Proposition 2, which is similar to the Kac inequalities.[2,8] Let us now consider the probability $P(x, t)$ to be at x at time t (starting from 0 at time $t = 0$). We will deal with its Laplace transform $\tilde{P}(x, E)$:

$$\tilde{P}(x, E) = \int_0^\infty P(x, t) \exp(-Et)\,dt \tag{10}$$

Since $P(x, t)$ satisfies Eq. (2), we have

$$\tilde{P}(x, E) = \sigma^{-2}(x)\, \sigma^2(0)\, G(0, x, E) \tag{11}$$

where G is the kernel of the resolvent (Green function) of the operator $H = 1/2\sigma^2\, \partial^2/\partial x^2$. The operator H has to be considered as an unbounded positive operator in $L^2(R, \sigma^{-2}\, dx)$ and is essentially self-adjoint on $C_0^\infty(R)$. Thus, G is well defined for $E > 0$. Furthermore, one is often interested in the recurrence times: in the case of a random walk on Z, the recurrence time (at 0) is defined as the first time t such that $X(t) = 0$ and $X(t') \neq 0$ for some $t' < t$. For a continuous walks the same definition would be meaningless, but if we are interested in the distribution of large recurrence times t we can as well define t^* as the first time such that $X(t) = 0$ starting from a (one could also consider the walks starting from $-a$; these two definitions may give rise to different laws for recurrence times). Now this definition makes sense in the continuous case: t^* is the Markovian stopping time at 0. Let us set $\tilde{R}^+(\mathrm{E}) = \mathsf{E}(\exp -Et^*)$: $\tilde{R}^+(E)$ is the Laplace transform of $R(a, 0, t)$ as defined in the introduction. Similarly, one can define $\tilde{R}^-(E)$ as the Laplace transform of $R(-a, 0, t)$.

In order to estimate $\tilde{R}^+(E)$ we remark that $G(0, X_t, E) \exp -Et$ is a (bounded) martingale [formally, it satisfies $(1/2\sigma^2\, \partial^2/\partial x^2 - \partial/\partial t) f = 0$], thus we have

$$G(0, a, E) = G(0, X_0, E) = \mathsf{E}(G(0, X_{t^*}, E) \exp -Et^*) = \tilde{R}^+(E)\, G(0, 0, E) \tag{12}$$

So $\tilde{R}^+(E)$ and $\tilde{P}(x, E)$ are given by $G(0, x, E)$; thus, we have only to estimate $G(0, x, E)$. It is well known that there exist two functions j^+ and j^- satisfying

$$(1/2\sigma^2\, \partial^2/\partial x^2 - E)\, j = 0 \tag{13}$$

which are respectively in $L^2([0, +\infty[, \sigma^{-2}\, dx)$ and in $L^2(]-\infty, 0], \sigma^{-2}\, dx)$. G can be expressed as

$$\begin{aligned} G(0, x, E) &= j^-(0)\, j^+(x)/W(j^+, j^-) \qquad \text{if} \quad x > 0 \\ &= j^-(x)\, j^+(0)/W(j^+, j^-) \qquad \text{if} \quad x \leqslant 0 \end{aligned} \tag{14}$$

where W is the Wronskian of j^+ and j^-: $W = 1/2\sigma^2(0)[j^-(0)\, j^{+\prime}(0) - j^{-\prime}(0)\, j^+(0)]$.

Let $M_x(X)$ be defined as $M(X)$ for a diffusion starting at x; then the equation $M_x(X) = t$ has clearly two solutions for any time: one is larger than x (say) and the other is smaller. Then the following Proposition holds:

Proposition 2 (Kac[8]). Let $X_x^+(t)$ and $X_x^-(t)$ be the distance from x of $M_x(X)=t$ (respectively larger and smaller than x); then j^+ and j^- satisfy

$$\begin{aligned} X_x^+(1/E)/2 < |j^+(x)/j^{+\prime}(x)| < X_x^+(1/E) \\ X_x^-(1/E)/2 < |j^-(x)/j^{-\prime}(x)| < X_x^-(1/E) \end{aligned} \tag{15}$$

Proof. In order to simplify the notations, let us prove (15) for $x=0$; the proof does not depend on this choice. One easily checks that j^+ cannot vanish (let us now suppose that j^+ is positive); then, by (13), j^+ is convex and thus $j^{+\prime}$ is negative and increases to 0 at $+\infty$. Thus we have

$$j(u)=j(0)+\int_0^u j'(y)\,dy>0$$

$$j'(u)=j'(0)+2E\int_0^u \sigma^{-2} j(y)\,dy<0$$

Thus,

$$j(0)\left(1+2E\int_0^u dy\int_0^y dz\,\sigma^{-2}\right)+uj'(0)>0 \tag{16}$$

$$j'(0)\left[1+2E\int_0^u \sigma^{-2}(y)\,dy\int_0^y dz\right]+2Ej(0)\int_0^u \sigma^{-2}(y)\,dy<0 \tag{17}$$

These inequalities are true for any $u>0$; choosing $u=X^+(1/E)$ in (16) provides the lower bound for (15). Integrating by parts we have

$$\int_0^u \sigma^{-2}(y)\,dy\int_0^y dz=u\int_0^u \sigma^{-2}(y)\,dy-\int_0^u dy\int_0^y \sigma^{-2}(z)\,dz \tag{18}$$

Thus, for the same choice of u, (17) and (18) provide the upper bound of the lemma.

By Eq. (14), the following propositions are direct consequences of Proposition 2:

Proposition 3. $\tilde{P}(0,E)$ [see Eq. (10)] satisfies

$$X^+X^-/(X^++X^-)<\sigma^2(0)\,\tilde{P}(0,E)<2X^+X^-/(X^++X^-)$$

Proposition 4. $\tilde{R}^+(E)$ [the Laplace transform of $R(0,a,t)$] satisfies [and similarly $R^-(E)$]

$$1-2a/X^+<\tilde{R}^+(E)<1-a/X^++2E\int_0^a dy\int_0^y \sigma^{-2}(z)\,dz$$

Proposition 5. $\tilde{P}(x, E)$ satisfies, for $x > 0$,

$$\sigma^2(0)\,\tilde{P}(0, E)\exp\left[-2\int_0^x 1/X_y^+(1/E)\,dy\right] < \sigma^2(x)\,\tilde{P}(x, E)$$

$$\sigma^2(x)\,\tilde{P}(x, E) < \sigma^2(0)\,\tilde{P}(0, E)\exp\left[-\int_0^x 1/X_y^+(1/E)\,dy\right]$$

and similarly for $x < 0$.

Remark 1. Obviously, as E goes to zero, the third term of the upper bound of Proposition 4 becomes negligible if there is no invariant measure.

Remark 2. In the above propositions, notice that the upper and lower bounds are equivalent and thus they should be sufficient for most practical problems. Nevertheless, the same techniques (Proposition 2) can be extended to higher orders to provide (exponentially fast) converging upper and lower bounds.

2.4. Bounds on $P(x, t)$

In this part we use the above propositions to get rough estimates on the equivalent time-dependent quantities. In the simplest cases one can invert the Laplace transform and get the time dependence of P and R (at least for large t) through Tauberian theorems. However, in the general case, the following propositions hold:

Proposition 6. $P(0, t)$ satisfies

$$P(0, t) < 1/t\tilde{P}(0, 1/t)$$

$$e/(1+e)\,\tilde{P}(0, 1/t) < \int_0^t P(0, s)\,ds < e\tilde{P}(0, 1/t)$$

Proposition 7. Let $S(t)$ be the probability that the first "recurrence" time is larger than t; then

$$e/(1+e)\,et[1 - \tilde{R}^+(0, 1/t)] < \int_0^t S(s)\,ds < et[1 - \tilde{R}^+(0, 1/t)]$$

Proof. The first upper bound in Proposition 6 relies on the convexity on P ensured by the spectral decomposition; then

$$\tilde{P}(0, E) = 1/E\int_0^t E\exp(-Es)\,P(0, s)\,ds$$

$E \exp(-Es)\, ds$ is a normalized measure and applying the Jensen inequality we get the first upper bound. The next upper bound in Proposition 6 is obvious by restricting the integral in (10) to the interval $[0, 1/E]$. Then, notice that $P(0, t)$ is decreasing, as can be seen from the spectral decomposition of the operator H. Thus,

$$\begin{aligned}\tilde{P}(0, E) &< \int_0^t P(0, s)\, ds + \exp(-Et)\, P(0, t)/E \\ &< \int_0^t P(0, s)\, ds\, [1 + \exp(-Et)/Et]\end{aligned}$$

This provides the lower bound of Proposition 6. The proof of Proposition 7 is the same, since $S(t)$ is decreasing.

Remark 1. Notice that in Proposition 6 the first inequality cannot be completed by a lower bound for $P(0, t)$: one can find examples where $P(0, t)$ decreases faster than this upper bound.

Remark 2. If the diffusion is limited to a finite interval with open boundaries, the function $M(x)$ diverges as x approaches a boundary. One easily checks that Propositions 2–7 extend. In this case X^+ is bounded; thus, by Propositions 3 and 6, $\int_0^t P(0, s)\, ds$ converges as t goes to ∞ and the motion is transient. Furthermore, by Proposition 4, $\tilde{R}^+(E)$ does not go to 1 as E goes to 0, which again proves that the motion is transient and provides the probability $1 - \tilde{R}^+(0)$ to go (starting from 1) to the right never crossing the origin. If the Itô diffusion comes from a general diffusion (with drift), this remark provides a simple criterion for the recurrence of the process.

2.5. Bounds for $\mathsf{E}(|X_t|)$

Proposition 1 can be inverted in the following sense:

Proposition 8. Let X^+ and X^- be defined as in Proposition 2; if infinity is not an entrance boundary ($|x|\, \sigma^2$ is not integrable at ∞), then

$$\begin{aligned}&e/2(1+e)\, X^+(t)\, X^-(t)/[X^+(t) + X^-(t)] \\ &\qquad \leqslant \mathsf{E}(|X_t|) \leqslant eX^+(t)\, X^-(t)/[X^+(t) + X^-(t)]\end{aligned}$$

Multiplying by $|x|$ both sides of Eq. (2) and integrating by parts, we get formally

$$d/dt\, \mathsf{E}(|X_t|) = 1/2\sigma^2(0)\, P(0, t) \tag{19}$$

Thus,

$$\mathsf{E}(|X_t|) = 1/2 \int_0^t \sigma^2(0)\, P(0, s)\, ds \tag{20}$$

The right-hand side of (20) is estimated in Proposition 6, which provides Proposition 8. It remains only to set rigorously Eq. (20).

Proof of Eq. (20). Let us define $f_\varepsilon(x)$ as

$$|x| - \varepsilon/2 \text{ if } |x| > \varepsilon, \qquad x^2/2\varepsilon \text{ if } |x| \leqslant \varepsilon \tag{21}$$

and let $\chi_\varepsilon(x)$ be the characteristic function of the interval $[-\varepsilon, \varepsilon]$. We choose some $t > 0$; then, if t_N is the stopping time defined in Section 2.2 (proof of Proposition 1), one has[1]

$$\mathsf{E}(f_\varepsilon(X_{t_N})) - \mathsf{E}\left(1/(2\varepsilon) \int_0^{t_N} \sigma^2(X_{t'})\, \chi_\varepsilon(X_{t'})\, dt'\right) = 0 \tag{22}$$

since $1/\varepsilon\, \chi_\varepsilon(x)$ is the second derivative of $f_\varepsilon(x)$. We first let N go to ∞: the contribution of the events such that $t_N < T$ vanishes if $N/M^2(N)$ goes to zero as N increases. As in Section 2.2, this occurs as soon as $\pm\infty$ are natural boundaries. In the second step, we let ε go to zero; there the regularity of $\sigma^2(x)\, P(x, t)$ for $t > 0$ ensures that (22) goes to

$$\mathsf{E}(|X_t|) = 1/2 \int_0^t \sigma^2(0)\, P(0, t')\, dt'$$

This ends the proof of Proposition 8.

3. APPLICATIONS

3.1. Diffusion Speed Behaving As a Power at Infinity

We suppose that $\sigma^2 \approx |x|^a$ at both infinities. There are three different cases:

(i) $a \leqslant 1$: there is no invariant measure and $M(x) \approx x^{2-a}$ [$x \ln(x)$ if $a = 1$]; thus, $\mathsf{E}(|X_t|) \approx t^{1/(2-a)}$ ($t/\ln t$ if $a = 1$).

(ii) $1 < a \leqslant 2$: there is an invariant measure and infinity is a natural boundary, $M(x) \approx x$, and thus $\mathsf{E}(|X_t|) \approx t$.

(iii) $2 < a$: there is an invariant measure and infinity is an entrance boundary, $M(x) \approx x$, but $\mathsf{E}(|X_t|)$ is bounded [it should converge to $\int_0^\infty |y|\, \sigma^{-2}(y)\, dy / \int_0^\infty \sigma^{-2}(y)\, dy$].

If σ^{-2} is exactly a power law (at least, say, for $|x| > 1$), the asymptotic solution is

$$P(x, t) \approx x^{-a}/t^{1-a/2-a} \exp(-cx^{2-a}/t) \tag{23}$$

This can be justified by direct spectral analysis involving Bessel functions. This simple case supports the physical idea that for arbitrary σ^2, $P(x, t)$ looks like $\sigma^{-2}P(0, t)$ multiplied by an exponential cutoff. $P(0, t)$ should be defined by the normalization condition:

$$\int_{-\infty}^{+\infty} P(x, t)\, dx = 1 \approx \int_0^{X^*} \sigma^{-2}(x)\, dx\, \sigma^2(0)\, P(0, t) \tag{24}$$

where X^* satisfying $M(X^*) = t$ is the typical abcissa reached within time t. The cutoff term can be estimate using Proposition 4. First, one easily checks that $X_x^+(E)$ behaves like $\sigma/E^{1/2}$ provided infinity is a natural boundary ($a < 2$); otherwise we have

$$X_x^+(E) \int_0^{\infty} \sigma^{-2}(x)\, dx \approx 1/E$$

Thus, in the former case the cutoff of the Laplace transform is $\exp(-E^{1/2} \int \sigma^{-1}\, dx)$. Since the inverse Laplace transform of $\exp(-E^{1/2}C)$ is $\exp(-2C/t)$, one expects that

$$P(x, t) \approx \sigma^{-2}(x)\, P(0, t) \exp\left[-2 \int_0^x \sigma^{-1}(y)\, dy/t\right] \tag{25}$$

This result fits (10) through Laplace transform. In the latter case, $1/X_x^+(1/E)$ is integrable; thus, there is no cutoff term (which clarifies the restrictions in Section 2.2) and the invariant measure is reached uniformly all over R:

$$P(x, t) \approx \sigma^{-2}(x) \tag{26}$$

Remark. In the previous estimates we have implicitly assumed that σ is regular (without large oscillations); let us now the opposite case, where the fluctuations of σ rule the behavior of the diffusion.

3.2. Random Diffusion Speed

Let us first consider the "regular" case where the random variable σ^{-2} can be averaged. In ref. 3, it is proved that $X(t)/\sqrt{t}$ converges in law to a Gaussian variable with variance $\sigma_{\text{eff}}^{-2} = \mathrm{E}(\sigma^{-2})$. Our estimates agree with this result, but our approach is quite different: for instance, Section 2.2 allows us to compute the moments of $X(t)$ (which is not strictly speaking the case in ref. 3). For example, $M(x)/x^2$ goes to σ_{eff}^{-2} by the ergodic theorem and thus $\mathsf{E}(X^2(t))$ goes almost surely to $\sigma_{\text{eff}}^2 t$. In the same way one can estimate the higher moments. Furthermore, $\tilde{P}(x, E)$ can be easily estimated, providing large fluctuations yet Gaussian.

Now, let us deal with the nonaveraging case. Let us assume that $\sigma(x, \omega)$ is a (translation invariant) random process such that σ^{-2} is not integrable. For instance, let us assume that σ^{-2} is piecewise constant on integer intervals and that the distribution of σ^{-2} on an interval is in the domain of attraction of a stable law of order a: $P(\sigma^{-2}) \approx c/(\sigma^{-2})^{1+a}$ for small σ with $0 < a < 1$.

Then $x^{-1/a} \int_0^x \sigma(y, \omega)^{-2}\, dy$ converges in law to a stable law of order a and so $X^{-1/a-1}M(X)$. This provides the diffusion exponent $d_w = 1 + 1/a$ and the behavior of $X^+(t) \approx t^{a/1+a}Y$. Thus, $\tilde{P}(0, E)$ behaves like $E^{-a/1+a}$ and by Tauberian theorem we get $P(0, t) \approx t^{a/1+a}$. Furthermore, the cutoff term in $\tilde{P}(x, E)$ can be estimated by noticing that $E^{1/2+a}/X_x^+(1/E)$ is now an averaging random variable; thus, the cutoff term in $\tilde{P}(x, E)$ should behave (by the ergodic theorem) like $\exp(-xE^{a/1+a})$, which in turn corresponds to a cutoff term $\exp(-x^{1+a}/t^a)$ for $P(x, t)$.

3.3. Diffusion in a Random Potential

We now go to another random model the random walk in a random medium, corresponding to the process

$$dX = -dV(X)/dx\, dt + dB \tag{27}$$

V is a random potential. This kind of system has been studied, for instance, in refs. 5–7 and 9. Sinaï[6] studies a discrete version of (27) where V is a random walk. He proves that $X(t)/\ln^2 t$ converges in law to a random variable depending only on V and t. This result has been adapted to the continuous version by Brox[7]. Kesten *et al.*[5] study the asymmetric models (with nonzero mean drift). In the following, we first restrict ourselves to the case without mean drift (where the law of V is even in V), but the further discussion can be done in the same way in the general case. Bouchaud *et al.*[9] consider the case[7] where V is a Brownian motion in x and show that the averaged (over V) probability transition $P(0, 0, t)$ behaves as

$$\langle P(0, 0, t) \rangle = C/\ln^2 t \tag{28}$$

Using results of Section 2, we will discuss the probabilistic meaning of this latter result. In particular, we will show that $P(0, 0, t)$ cannot be characterized by its mean value, and we will give phenomenological laws for this quantity.

For general V, the change of variables

$$dy = \exp[V(x)]\,dx \tag{29}$$

provides for Y an Itô diffusion $dY = \sigma(Y)\,dB$, where $\sigma(y) = \exp[V(x)]$. Results of Section 2 can be readily expressed in the x variable

$$1/t \int_0^t P(0, s)\,ds \approx \inf(Y^+, Y^-)/t \tag{30}$$

where $Y^+(t) = \int_0^{X^+} dx \exp V(x)$, and, as previously, $X^+(t)$ $[X^-(t)]$ is the positive (negative) solution of

$$M(X) = \int_0^X dx \exp V(x) \int_0^x dx' \exp -V(x') = t \tag{31}$$

First, let us remark that if V is a stationary process such that $\langle \exp V \rangle$ is finite, by the ergodic theorem we have

$$M(X^+) \approx X^{+2} \langle \exp V \rangle \langle \exp -V \rangle \tag{32}$$

This indicates that the diffusion is normal, with a diffusion coefficient

$$\sigma_{\text{eff}}^{-2} = \langle \exp V \rangle \langle \exp -V \rangle$$

Let us suppose from now on that V is a Brownian process in x. As above, we only provide heuristic ideas in order to obtain the main features of this diffusion. These ideas can be made more precise using the properties of the "depressions" introduced in ref. 7. As V is unbounded, the integrals determining M and Y are dominated by the largest values of the exponentials. Then, around an extremum of V, the behavior of $V(x)$ does not depend on the total integration range X^+, nor on the value of this extremum. Thus, we do not get any "prefactor" and the typical behavior of Y and M is

$$Y(X) \approx \exp[\sup_{X > x > 0} V(x)] \tag{33}$$

Similarly,

$$M(X) \approx \exp[\sup_{X > x > x_0 > 0} V(x) - V(x_0)] \tag{34}$$

That is, $M(X)$ is determined by the "depressions"[7] in $[0, X]$. Then, we can rewrite (30)

$$\int_0^t P(0, s)\, ds/t \approx t^{\eta - 1} \tag{35}$$

where

$$\eta = \mathrm{Min}(\eta^+, \eta^-)$$

$$\eta^+ = \sup_{X^+ > x > 0} V(x) / \sup_{X^+ > x > x_0 > 0} V(x) - V(x_0)$$

$$\eta^- = \sup_{X^- < x < 0} V(x) / \sup_{X^- < x < x_0 < 0} V(x) - V(x_0)$$

We now argue that η is a random variable (such that $0 < \eta < 1$) which has a law independent of t from the scaling properties of the Brownian motion. Then we get for the quantity $P(0, t)$, which physically does not have to be averaged on the random potential, an extremely dispersed law. We could, for example, define an exponent γ by

$$\gamma = -\langle \ln P \rangle / \ln t = \langle \mathrm{Min}(\eta^+, \eta^-) \rangle$$

We will not attempt here to compute γ; let us only recover the result of Bouchaud *et al.* for the averaged quantity $\langle P(0, t) \rangle$. This quantity is dominated by the behavior of the law of η near $\eta = 1$. The probability that $\eta > 1 - \varepsilon$ with ε small is just the probability that $V(x)$, starting form 0, stays above $-\varepsilon V$ before reaching V at $x = X$, thus, the probability that the exit out of $[-\varepsilon V, V]$ is achieved at V. Since $V(x)$ is a martingale, this quantity is ε. Taking now into account that $\eta = \mathrm{Min}(\eta^+, \eta^-)$, we get

$$\left\langle \int_0^t P(0, s)\, ds/t \right\rangle \approx \int_0^\varepsilon \int_0^\varepsilon d\varepsilon\, d\varepsilon' \exp[-\ln t\, \mathrm{Max}(\varepsilon, \varepsilon')] \approx (\ln t)^{-2} \tag{36}$$

In fact, a rigorous lower bound on $\langle 1/t \int_0^t P(0, s)\, ds \rangle$ can be easily obtained by considering the above special samples of V without introducing the exponent η. Indeed, for such samples the diffusion is trapped near the origin, which provides the lower bound. However, for typical samples, the trapping occurs at a distance $d(t)$ about $(\ln t)^2$ from the origin and $P(0, t)$ behaves like $\exp[-V(d)]$ as the invariant measure is almost reached in the interval $[0, d]$.

We now briefly study the typical Laplace transform $P(x, E)$. The exponential factor occurring in Proposition 5, $\int_0^y dz/Y^+(1/E)$, can be estimated using the same techniques as above; this yields

$$\int_0^y dz/Y^+(1/E) = \int_0^x dx\, E^{\eta(x)} \tag{37}$$

where η has exactly the same definition as above, but starting from x instead of 0. From the ergodic theorem and the above estimates of the law of η near 0, we get

$$\int_0^X dx\, E^{\eta(x)} \approx X/\ln(1/E) \tag{38}$$

for X large with respect to $(\ln t)^2$ in order to ensure the convergence to the mean. This indicates a law of rare events (for large x)

$$P(x, t) \approx \exp[-V(x) - x/\ln t] \tag{39}$$

The first term in (39) is the invariant measure and corresponds to σ^{-2} in Proposition 5 multiplied by the Jacobian of the change of variable (29). Notice that the obtained scaling is not the natural one $x/(\ln t)^2$. This comes from the fact that the convergence to the mean in (38) occurs on a scale much larger than $(\ln t)^2$.

Furthermore, if one adds to $V(x)$ in (27) a constant drift μ (which is the case in refs. 59), the same fluctuations occur for $P(0, t)$. Bouchaud *et al.*[9] show that $\langle P(0, 0, t)\rangle \approx t^{-\mu}$ even if $\mu < 1$. On the other hand, $P(0, 0, t)$ is almost surely integrable with respect to t, as can be seen from Remark 2 after Proposition 6, by noticing that $y(x)$ defined by

$$dy = \exp[V(x) - \mu x]\, dx$$

is almost surely bounded (by the strong law of large numbers).

We conclude this section by noting that the same analysis can be done if $V(x)$ is not a Brownian motion. Modifications then arise from the behavior of $Y(X)$ in which prefactors can occur.

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